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Some of My Favorite Problems[1]

Remark. I want to stress that being good at mathematics competitions is not necessary for success in mathematics research. I know many people who did really well in their PhD and are continuing to do great work, without having excelled at contests. I think that finding math enjoyable and exciting goes a long way. Sometimes, one can lose these qualities by becoming too focussed on performance.

Problem 1

Using one straight cut, divide a regular tetrahedron into two identical pieces. Note that the two pieces should be identical, rather than identical up to a reflection.

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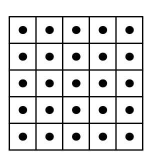
Problem 2

You place 1 coin on each square of a 5×5 checkerboard. Then you move *every* coin to some adjacent square (up, down, left or right; diagonal squares *are not* adjacent). Show that, no matter how this is done, there will be two coins that will end up on the same square.



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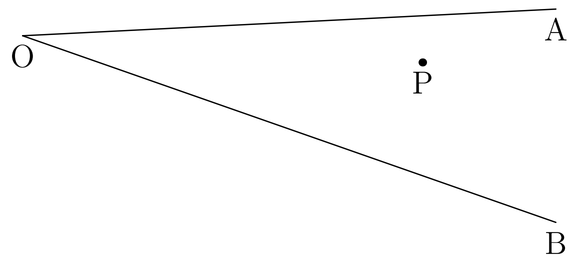
[1] These should all be challenging. Some of the particularly difficult ones are marked by an asterisk.

Problem 3 *

Starting from the figure below and using a straight-edge and compass, construct a right-angled *isosceles* triangle ΔQPR such that:



- Q lies on the line OA,
- R lies on the line OB,
- $\angle QPR$ is a right angle.

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Problem 4 *

Find all triples of natural numbers that solve the equation:

$$7^a + 3^b = c^2$$

. . . .

Problem 5

Here n will be a natural number.

1. Show that $n^3 - n$ is always divisible by 3.
2. Show that $n^5 - n$ is always divisible by 5.
3. Show that $n^7 - n$ is always divisible by 7.
4. Show that $n^{11} - n$ is always divisible by 11.
5. Show that $n^{13} - n$ is always divisible by 13.

Observe that $n^9 - n$ does not need to be divisible by 9. For example $2^9 - 2 = 510$, which is not a multiple of 9.

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Problem 6

1. You have 20 coins, one of which is fake. It is lighter than the others. Using a pan balance, find a way to determine, using no more than 3 weighings, the fake coin.

2. * You have 12 coins, one of which is fake. Its weight differs from the others, but you do not know if it is lighter or heavier. Using a pan balance, find a way to determine, using no more than 3 weighings, the fake coin.

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Problem 7

There is a road that stretches from one town to another, At some point in the morning, two people *simultaneously* depart from opposite ends of the road and start walking toward each other along this road. Each walks at constant speed, and their speeds are different. The two people pass each other exactly at 12 PM and continue walking. Then, one of them reaches the end of the road at 4 PM; the other does so only at 9 PM. At what hour in the morning did they depart?

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Problem 8

A square in the Cartesian plane has its vertices at the points:

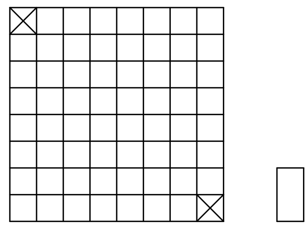
$$(-1,1) \quad (?,?) \quad (1,11) \quad (?,?)$$

in that order, as you go clockwise around the square. Fill in the missing coordinates.

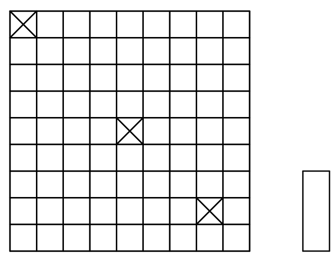
. . . .

Problem 9

1. Given a 8×8 checkerboard, delete two opposing corner squares. Why is it impossible to cover the remaining squares with non-overlapping 2×1 dominoes? (Each 2×1 domino is a 2×1 rectangle that completely covers two adjacent squares.)



2. * Now, given an 9×9 checkerboard, we delete any three squares on the main diagonal. Show that it is impossible to cover the remaining squares with non-overlapping 3×1 dominoes.



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Problem 10 *

Suppose that every point of the plane is coloured either red, blue or green. Regardless of how this is done, show that you can always find two points of the same colour that are exactly one centimeter apart.

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Problem 11

There are 20 prisoners, who are ordinarily kept in complete isolation from one another. There is an empty room in the prison. The room contains nothing but a broken light switch that is either in the "on" or "off" position. One day, the warden decides that every morning, he will choose one prisoner and take this prisoner to this empty room. Furthermore, he decides that he will let all prisoners go free when one of them can prove to him conclusively that each of the other prisoners has visited the room at least once. The warden himself does not go inside the room. The prisoners have one day to agree on a strategy, after which they will again be kept in complete isolation from one another.

Find a strategy that will allow the prisoners to eventually go free. You can assume the light switch is initially in the "off" position. (After you've solved this problem, try to do it without this assumption.)

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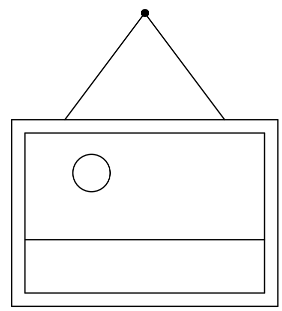
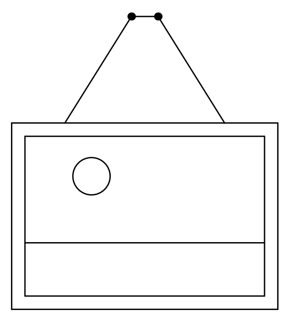
Problem 12 *

Here is a schematic picture of a painting hanging from a nail on the wall. If you pull



the nail out, the painting will fall. Now suppose that there are two nails. It is easy to hang the painting in such a way that pulling either nail out will *not* cause the painting to fall, as long as the other nail stays in place:



Try hanging the painting in such a way so that pulling *either* nail out *will* cause the painting to fall.

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Problem 13

Imagine that you have a clock whose minute hand and hour hand look exactly the same. Usually, you can still figure out what time it is by glancing at this clock. But there are some moments in the day when this is not possible. Exactly *how many* such moments are there in one day?

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Problem 14

Choose 8 distinct numbers from 1 to 100 at random. How many ways are there to make this choice?

In the remaining parts of this problem, we will go through all possible choices and calculate average values over all choices.

1. Show that the average value (over all possible choices) of the mean of the 8 numbers is $\frac{101}{2}$.

2. Show that the average value (over all possible choices) of the minimum of the 8 numbers is $\frac{101}{9}$

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Additional Comment. Thanks to my Dad, I owned a copy of the "USSR Olympiad Problem Book" by Shklarsky, Chentzov, and Yaglom (in English). It's a great collection of challenging problems. Nowadays, it is easy to find a PDF version online.

Feel free to reach out to me if you would like suggestions, or have questions about the above problems.

The problems from my talk

1. Let $a < b < c$ be the roots of the equation

$$x^3 - 6x^2 + 5x - 1$$

in increasing order.

   (a) Determine the value of $a^3 + b^3 + c^3$ and the value of $a^5 + b^5 + c^5$.

   (b) * Determine with proof which number is closer to being an integer: c^{2003} or c^{2004}.

2. * Prove that $100! + 1$ is divisible by 101.