Decomposing measures on modules

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Abstract

In this note, motivated by random matrix theory, we study measures on a finite R-module M, where R is a finite local ring. We show that the vector space of these measures admits an orthogonal decomposition, whose components are parametrized by homomorphisms from M to the dualizing module of R. This can be regarded as a slight generalization of the usual Fourier decomposition.

1 Introduction

We would like to slightly generalize the Fourier decomposition of measures on finite abelian groups to the case of modules over a ring. First, we give a rephrasing of the usual Fourier decomposition.

Denote $\mathbb{Z}/m\mathbb{Z}$ as $\mathbb{Z}_m$. Let G be a finite $\mathbb{Z}_m$-module, in other words, G is a finite abelian group that is m-torsion.

Definition. Let $\mathcal{P}(G)$ be the vector space of signed measures on G.

By Fourier analysis, there is an orthogonal decomposition of $\mathcal{P}(G)$ into 1-dimensional subspaces indexed by elements of

$$\mathrm{Hom}(G, \mathbb{C}^*) \cong \mathrm{Hom}(G, \mathbb{Z}_m)$$

. We can group these subspaces together into those parametrized by "isomorphic" homomorphisms, i.e. those homomorphisms that differ by an automorphism of $\mathbb{Z}_m$. This gives a decomposition of $\mathcal{P}(G)$ into orthogonal subspaces, indexed by

$$\chi \in \mathrm{Hom}(G, \mathbb{Z}_m) \big/ \mathbb{Z}_m^*.$$

These subspaces, which we denote as $V(\chi)$, can be uniquely characterized by the following three properties, without any reference to Fourier analysis:

(A) The pushforward map, induced by any homomorphism in the equivalence class χ, is injective on $V(G, \chi)$. Equivalently, the elements of $V(G, \chi)$ are constant on $ker(\chi)$-cosets.

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(B) $V(G,\chi)$ and $V(G,\chi')$ are orthogonal unless $\chi = \chi'$.

(C) The subspaces $V(G,\chi)$ span $\mathcal{P}(G)$, i.e.

$$\mathcal{P}(G) \cong \bigoplus_{\chi} V(G,\chi).$$

The present paper. In this note, M will be a finite module over a finite local ring R. We will define and study a decomposition of $\mathcal{P}(M)$ that shares similar properties to the decomposition in the previous paragraph.

Theorem 1.1. *Let R be a finite local ring, let M be a finite R-module and denote by ω the dualizing module of R. To every*

$$\chi \in Hom_R(M,\omega)\Big/R^* \tag{1.1}$$

we can associate a vector space $V(M,\chi) \in \mathcal{P}(M)$ with the following properties:

(A) The pushforward map induced by any homomorphism in the equivalence class χ, is injective on $V(M,\chi)$. In other words, the elements of $V(M,\chi)$ are constant on $ker(\chi)$-cosets.

(B) $V(M,\chi)$ and $V(M,\chi')$ are orthogonal unless $\chi = \chi'$.

(C) The vector spaces $V(M,\chi)$ span $\mathcal{P}(M)$, i.e.

$$\mathcal{P}(M) \cong \bigoplus_{\chi} V(M,\chi). \tag{1.2}$$

The main practical objective of this paper is to prove Theorem 1.1 and to establish the useful inequalities (10.1) and (11.1). Theorem 1.1 is a consequence of Corollary 8.1.1. The inequality (10.1), proven in §10, is a straightforward consequence of the Cauchy-Schwarz inequality and the above decomposition. Along the way, we also establish some additional properties of the vector spaces $V(M,\chi)$.

Outline. We start by defining a decomposition of $\mathcal{P}(M)$ into orthogonal subspaces $V(M,N)$ where N runs through all submodules of M. This part is purely formal. Then we show, using some basic facts in commutative algebra, that $V(M,N)$ is non-zero if and only if $N \cong ker(\chi)$ for some χ of the form (1.1). The subspaces $V(M,\chi)$ are defined as $V(M,ker(\chi))$. In §9, we study a coarsening of the decomposition (1.2). §10 is devoted to the inequality (10.1).

2 Preliminaries: measures on a module

Throughout, R will be a finite local ring and M will be a finite R-module. As in the introduction, we will denote by $\mathcal{P}(M)$ the real vector space of signed measures on M.

Define

$$\Big\langle \cdot, \cdot \Big\rangle_M : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathbb{R} \tag{2.1}$$

to be the Euclidean inner product on $\mathcal{P}(M)$, regarded as $\mathbb{R}^{(\#M)}$.

For any submodule N of M, let $\mathcal{P}(M,N) \subset \mathcal{P}(M)$ denote the space of signed measures on M that are constant on N-cosets.

We make the following remarks:

- The vector space $\mathcal{P}(M,N)$ is isomorphic to the space of measures on M/N, i.e.
$$\mathcal{P}(M,N) \cong \mathcal{P}(M/N)$$
- If $N_2 \subset N_1$,
$$\mathcal{P}(M,N_1) \subset \mathcal{P}(M,N_2)$$
.
- We can also define a map in the opposite direction. Indeed, we can take the adjoint of the inclusion map $\mathcal{P}(M,N_1) \to \mathcal{P}(M,N_2)$ with respect to the inner product (2.1), to get a map that we denote as proj_{N_1,N_2}:

$$\mathrm{proj}_{N_1,N_2} : \mathcal{P}(M,N_2) \to \mathcal{P}(M,N_1).$$

Properties of proj proj_{N_1,N_2} can also be defined as follows: given a measure in $\mathcal{P}(M,N_2)$, "average out" this measure over N_1. This definition does not depend on N_2 and it is defined on all of $\mathcal{P}(M)$. Hence we will subsequently simply write proj_{N_1}.

We note the following properties of proj that follow from the definition and the preceding discussion:

- The restriction of proj_N to $\mathcal{P}(M,N)$ is the identity.
- $\mathrm{proj}_{N_1}\mathrm{proj}_{N_2} = \mathrm{proj}_{N_1+N_2}$.

3 Decomposition of $\mathcal{P}(M)$ into orthogonal subspaces

Definition. Define

$$V(M,N) \stackrel{\text{def}}{=\!=} \bigcap_{\substack{N\subset N' \\ N\neq N'}} \ker\left(\mathcal{P}(M,N) \xrightarrow{\mathrm{proj}_{N',N}} \mathcal{P}(M,N')\right)$$

Lemma 3.1. *$V(M,N_1)$ and $V(M,N_2)$ are orthogonal subspaces of $\mathcal{P}(M)$ for $N_1 \neq N_2$.*

Proof. Suppose $\nu_1 \in V(M,N_1)$ and $\nu_2 \in V(M,N_2)$ and $N_1 \neq N_2$. Then we have:

$$\left\langle \nu_1, \nu_2 \right\rangle = \left\langle \mathrm{proj}_{N_1}\nu_1, \nu_2 \right\rangle = \left\langle \mathrm{proj}_{N_2}\mathrm{proj}_{N_1}\nu_1, \nu_2 \right\rangle =$$

$$= \left\langle \mathrm{proj}_{N_1+N_2}\nu_1, \nu_2 \right\rangle = \left\langle \mathrm{proj}_{N_1+N_2}\nu_1, \mathrm{proj}_{N_1+N_2}\nu_2 \right\rangle = 0$$

because either ν_1 or ν_2 lies in the kernel of $\mathrm{proj}_{N_1+N_2}$. □

Lemma 3.2.

$$\mathcal{P}(M) = \bigoplus_{N \subset M} V(M, N)$$

Proof. In order to prove the lemma, it is sufficient to show that $\mathcal{P}(M)$ is spanned by the vector spaces $V(M, N)$. We prove the lemma by induction on $\#M$.

Base case. For the base case, assume $\#M = 1$. Then the statement says that the 1-dimensional space $\mathcal{P}(0)$ is spanned by the 1-dimensional space $V(0, 0) \cong \mathcal{P}(0)$, which is true.

Induction step. Now suppose that M is an R-module of cardinality $\#M > 1$ and suppose that the statement is true for all modules of cardinality less than $\#M$.

In particular, the statement of the lemma is true for all modules M/L, assuming that L is not trivial. Hence we can assume:

$$\mathcal{P}(M/L) = \bigoplus_{N \subset M/L} V(M/L, N).$$

By the identification,

$$\mathcal{P}(M/L) \cong \mathcal{P}(M, L)$$

it also follows that:

$$\mathcal{P}(M, L) = \bigoplus_{L \subset N \subset M} V(M, N). \tag{3.1}$$

We conclude the proof using the following lemma, which is a direct consequence of the definition of proj as an adjoint operator:

Lemma 3.2.1.

$$\ker\left(\mathcal{P}(M) \xrightarrow{proj_N} \mathcal{P}(M, N)\right)$$

is the orthogonal complement of $\mathcal{P}(M, N)$ *in* $\mathcal{P}(M)$.

Now it follows from the lemma that $V(M, 0)$ is the orthogonal complement in $\mathcal{P}(M)$ of the vector space:

$$span\left\{\mathcal{P}(M, N)\middle|N \neq 0\right\} \tag{3.2}$$

Hence every element in $\mathcal{P}(M)$ can be expressed as a sum of an element of $V(M, 0)$ and an element of (3.2). But, by (3.1), every element in (3.2) lies in the span of the vector spaces $V(M, N)$. Hence $\mathcal{P}(M)$ is also spanned by the vector spaces $V(M, N)$. $\square$

Other properties of $V(M, N)$ We list some other properties of the vector spaces $V(M, N)$:

- $V(M, N)$ is the subspace of $\mathcal{P}(M, N)$ that is orthogonal to $\mathcal{P}(M, N')$ for all N' that strictly contain N.
- $V(M, N) \cong V(M/N, 0)$.
- The vector spaces $V(M, N)$ and $\mathcal{P}(M, N)$ are invariant under translation by any element of M.

4 Fourier modules

In general, many of the vector spaces $V(M, N)$ are trivial. We are interested in identifying the non-trivial $V(M, N)$. As $V(M, N) \cong V(M/N, 0)$, this amounts to describing those modules L for which $V(L, 0) \neq 0$.

Definition. We say that a module L is *Fourier* if $V(L, 0)$ is not 0.

Hence,

$$\mathcal{P}(M) = \bigoplus_{\substack{N \subset M \\ M/N \text{ is Fourier}}} V(M, N) \tag{4.1}$$

We will wish to describe all Fourier modules over R.

Lemma 4.1. *If L is Fourier, and L' is a sub-module of L, then L' is Fourier.*

Proof. Indeed, suppose that L is Fourier and suppose that L' is a submodule of L.

Claim. *There exists a non-zero element $\nu \in V(L, 0)$ such that the restriction of ν to L',*

$$\nu|_{L'},$$

is non-zero.

> Indeed, $V(L, 0)$ contains a non-zero element because L is Fourier. Because $V(L, 0)$ is translation invariant, we can translate this element so that its restriction to L' is non-zero.

But ν lies in $\ker(\mathrm{proj}_N)$ for all N. In particular, ν lies in $\ker(\mathrm{proj}_N)$ for all $N \subset L'$. It follows that $\nu|_{L'}$ is a non-zero element of $V(L', 0)$. Therefore, L' is Fourier. $\square$

Lemma 4.2. *If a module L has a unique non-zero minimal submodule, then L is Fourier.*

Proof. Denote the minimal non-zero submodule as N_0. There exist elements of $\mathcal{P}(L)$ that are not constant on N_0-cosets. Therefore $\mathcal{P}(M)$ is not contained in $\mathcal{P}(L, N_0)$. Therefore, because N_0 is minimal, $\mathcal{P}(L)$ is not contained in the span of

$$\mathcal{P}(L, N) \qquad N \neq 0 \tag{4.2}$$

Therefore, the orthogonal complement of (4.2) in $\mathcal{P}(L)$ is non-empty. Hence, by Lemma 3.2.1, and the definition of $V(\,\cdot\,,\,\cdot\,)$, $V(L, 0)$ is non-empty and L is Fourier.

$\square$

5 Fourier modules that are powers of k

Recall that $k \stackrel{\text{def}}{=\!=} R/\mathfrak{m}$ is the residue field of R.

Theorem 5.1. *A module k^n is a Fourier module if and only if $n = 0$ or $n = 1$.*

In the rest of this section, we prove Theorem 5.1. First of all, 0 is always a Fourier module. The rest of the theorem will be proven by showing Lemma 5.2 and Lemma 5.3.

Lemma 5.2. *$V(k,0)$ has dimension $\#k-1$. In particular k is Fourier.*

Proof. We show the lemma using

$$\mathcal{P}(k) \cong V(0,0) \oplus V(k,0)$$

and comparing dimensions. $V(0,0) \cong \mathcal{P}(0)$ has dimension 1. $\mathcal{P}(k)$ has dimension $\#k$. Therefore $V(k,0)$ has dimension $\#k-1$. □

Lemma 5.3. *k^n is not a Fourier module over k, for any $n>1$.*

Proof. Let $n>1$. Again we use the decomposition:

$$\mathcal{P}(k^n) \cong \bigoplus_{N \subset k^n} V(k^n, N)$$

Recall that $V(k^n, N) \cong V(k^n/N, 0)$. Comparing dimensions, we find

$$(\#k)^n = \dim \mathcal{P}(k^n) = \sum_{i=0}^{i=n} \#\{N \subset k^n | k^n/N \cong k^i\} \dim V(k^i, 0)$$

It is now sufficient to show that

$$(\#k)^n = \sum_{i=0}^{i=1} \#\{N \subset k^n | k^n/N \cong k^i\} \dim V(k^i, 0) \tag{5.1}$$

But (5.1) can be rewritten as:

$$1 + (\#k-1)\#\{N \subset k^n | k^n/N \cong k\}$$

Let $\#Sur(k^n,k)$ denote the number of surjective homomorphisms from k^n to k and let $\#Aut(k)$ denote the number of automorphisms of k as an R-module. The preceding expression becomes:

$$1 + (\#k-1)\frac{\#Sur(k^n,k)}{\#Aut(k)} = 1 + \#Sur(k^n,k) = (\#k)^n$$

□

This concludes the proof of Theorem 5.1.

6 Fourier modules over R

Recall that R is a finite local ring with residue field k. In this section, we will determine all Fourier modules over R. We will need the fact that every finite local ring has a dualizing module [Eis95, Proposition 21.2]. We denote the dualizing module of R as ω.

Theorem 6.1. *Every Fourier module over R is a submodule of ω.*

Proof. The zero module is Fourier. Now suppose M is a non-zero Fourier module. By Lemma 4.1, every submodule of M must be a Fourier module. In particular

$$Hom(k, M)$$

is a Fourier module. This module is non-zero because M is non-zero, and it is isomorphic to k^n for some n. By Theorem 5.1,

$$Hom(k, M) \cong k. \tag{6.1}$$

Now, for an R-module L denote $Hom(L, \omega)$ as $D(L)$. ω is the dualizing module, hence $D(\cdot)$ is a dualizing functor. Therefore, (6.1) implies:

$$Hom(D(M), D(k)) \cong k \tag{6.2}$$

But $D(k) \cong k$, hence (6.2) implies

$$D(M) \otimes k \cong k$$

It is now a consequence of Nakayama's lemma that $D(M) \cong R/I$ for some ideal $I \subset R$. Therefore,

$$M \cong D(D(M)) \cong D(R/I) \cong Hom(R/I, \omega).$$

Therefore M is a submodule of ω. □

In the sections that follow, we denote $Hom(R/I, \omega)$ as ω_I.

Remark. We have shown that every Fourier module must be a submodule of ω. The converse is also true. Indeed, ω has a minimal non-zero submodule. Therefore, by Lemma 4.2, ω is Fourier, and by Lemma 4.1, all submodules of ω are Fourier.

7 Properties of ω_I

In this section, we list some standard properties of ω_I:

(A) $I \to \omega_I$ is an inclusion-reversing bijection between submodules of ω and submodules of R.

(B) $I = ann(\omega_I)$.

(C) $|\omega_I| = |R/I|$.

(D) The inclusion $R/I \hookrightarrow Hom(\omega_I, \omega_I)$ is surjective.

(E) ω_I is the dualizing module for the finite ring R/I.

We also prove the following lemma:

Lemma 7.1. *No two distinct submodules of ω are isomorphic.*

Proof. The submodules of ω are in bijection with submodules of R. Thus, we must prove that if

$$\omega_I \cong \omega_{I'},$$

then $I = I'$. But $\omega_I \cong \omega_{I'}$ implies that $ann(\omega_I) \cong ann(\omega_{I'})$ and hence $I = I'$. □

8 Relation with usual Fourier analysis

We can recast the decomposition of $\mathcal{P}(M)$ into the form (1.2), given in the introduction.

First of all, we introduce an equivalence relation:

Definition. Suppose $\chi, \chi' \in Hom(M, \omega)$. Then we write $\chi \sim \chi'$ if and only if $im(\chi) \cong im(\chi')$.

This allows us to rewrite the decomposition (4.1) as

$$\mathcal{P}(M) \cong \bigoplus_{\chi \in Hom(M,\omega)/\sim} V(M, ker(\chi)). \tag{8.1}$$

Lemma 8.1. *Suppose $\chi, \chi' \in Hom(M, \omega)$ and $\chi \sim \chi'$. Then there exists $r \in R^*$ such that $\chi' = r\chi$.*

Corollary 8.1.1. Hence, we can rewrite (8.1) as:

$$\mathcal{P}(M) \cong \bigoplus_{\chi \in Hom(M,\omega)/R^*} V(M, \chi) \tag{8.2}$$

where we write $V(M, \chi)$ to denote $V(M, ker(\chi))$.

Proof. (of Lemma 8.1) $im(\chi)$ and $im(\chi)$ are submodules of ω. If they are isomorphic, then by 7.A, they must be the same submodule of ω, say ω_I. Hence there exists an isomorphism σ of ω_I such that $\chi' = \sigma\chi$. But by 7.D, the only homomorphisms from ω_I to ω_I are given by multiplication by R/I. Hence, the only isomorphisms from ω_I to ω_I are given by multiplication by ${}^{R}/{}_{I}{}^{*}$, or alternatively by multiplication by R^*. Therefore σ must be of this form. □

Remark. We note again that χ is an equivalence class of homomorphisms. The number of homomorphisms in the equivalence class is determined by $im(\chi)$. By the preceding proof, if $im(\chi) = \omega_I$, then the number of elements in the equivalence class is $|{}^{R}/{}_{I}{}^{*}|$.

***The dimension of $V(M, \chi)$** The results in the remainder of this section are not necessary for the sequel. We include them for completeness.

First, we recall that

$$V(M, \chi) = V(M, ker(\chi)) \cong V(M/ker(\chi), 0) \cong V(im(\chi), 0)$$

and $im(\chi)$ is of the form ω_I for some I.

Lemma 8.2.

$$dim\, V(\omega_I, 0) = |{}^{R}/{}_{I}{}^{*}|$$

Proof. This can be proven by induction on $|{}^{R}/{}_{I}|$. Firstly, the statement holds for the maximal ideal because, by Lemma 5.2,

$$\dim V(k, 0) = \#k - 1.$$

For the induction step, we note that

$$\mathcal{P}(R/I) = \bigoplus_{\chi \in Hom(\omega/I,\omega)/R^*} V(\omega, ker(\chi))$$

Hence,

$$\dim \mathcal{P}(R/I) = \sum_J \frac{\#\{\chi \in Hom(\omega_I, \omega) | im(\chi) = \omega_J\}}{|{}^R/_{J^*}|} \dim V(\omega_J, 0)$$

The image of ω_I is either isomorphic to ω_I or has cardinality strictly smaller than ω_I. Using the inductive hypothesis, we can therefore conclude that the sum on the right can be rewritten as:

$$\frac{\#\{Hom(\omega_I, \omega) | im(\chi) = \omega_I\}}{|{}^R/_{I^*}|} \dim V(\omega_I, 0) + \\ + \sum_{J \neq I} \#\{Hom(\omega_I, \omega) | im(\chi) = \omega_J\} \tag{8.3}$$

whereas the left hand side is

$$\#\omega_I = \#Hom(\omega_I, \omega) = \\ = \sum_J \#\{\chi \in Hom(\omega_I, \omega) | im(\chi) = \omega_J\} \tag{8.4}$$

Comparing (8.3) and (8.4), we find that we must have

$$\dim V(\omega_I, 0) = |{}^R/_{I^*}|$$

□

Comparison with the classical case Suppose that we have a $\mathbb{Z}_{p^N}$ module G. Then, the usual Fourier decomposition decomposes $\mathcal{P}(G)$ into one-dimensional components parametrized by $Hom(G, \mathbb{C}^*) \cong Hom(G, \mathbb{Z}_{p^N})$. If we group together the components corresponding to homomorphsms that have the same kernel, then we recover the decomposition (8.2):

$$\mathcal{P}(G) \cong \bigoplus_\chi V(G, ker(\chi))$$

We verify that in this case, the dimension of $V(G, ker(\chi))$ is $\frac{p-1}{p} im(\chi)$, which coincides with $\left|\mathbb{Z}^*_{p^m}\right|$, where p^m is the annihilator of $im(\chi)$.

9 Isotypic Fourier components

In this section we will be interested in the space spanned by a certain natural subset of the $V(M, N)$. Namely, let us choose a Fourier module. This module is necessarily of the form ω_I. We are interested in explicitly describing

$$\bigoplus_{\substack{N \subset M \\ M/N \cong \omega_I}} V(M, N) \tag{9.1}$$

Remark. If $M/N \cong \omega_I$, then M/N is annihilated by I. Therefore, $IM \subset N$. It follows that

$$\bigoplus_{\substack{N \subset M \\ M/N \cong \omega_I}} V(M,N) \in \mathcal{P}(M, IM).$$

In fact, this is all we need for the sequel. In the rest of this section, for completeness, we give a more precise description of the vector space (9.1).

To formulate our theorem, we need some preliminary definitions:

***The spaces $W(M, IM)$** As mentioned in the preceding paragraph, the rest of this section is not necessary for the sequel.

Definition. Define $\mathcal{P}(M, IM)$ as before, as the set of measure on M that are constant on IM-cosets. Let $W(M, IM) \subset W(M, JM)$ be the space of signed measures on $\mathcal{P}(M)$ such that:

- Each element of $W(M, IM)$ is constant on IM-cosets.
- Each element of $W(M, IM)$ lies in the orthogonal complement of $\mathcal{P}(M, JM)$ for all ideals J that strictly contain I.

Lemma 9.1. *$W(M, IM)$ and $W(M, I'M)$ are orthogonal if $I \neq I'$.*

Proof. The proof is analogous to the proof of Lemma 3.1. Define proj_{JM} to be the operation that averages a measure on M over JM-cosets. Suppose that $w \in W(M, IM)$ and $w' \in W(M, I'M)$. We have

$$\left\langle w, w' \right\rangle = \left\langle \mathrm{proj}_{IM} w, w' \right\rangle = \left\langle \mathrm{proj}_{I'M} \mathrm{proj}_{IM} w, w' \right\rangle =$$

$$= \left\langle \mathrm{proj}_{(I+I')M} w, w' \right\rangle = \left\langle \mathrm{proj}_{(I+I')M} w, \mathrm{proj}_{(I+I')M} w' \right\rangle$$

The last expression must be 0 unless $I = I'$, by the same argument as in the proof of Lemma 3.1. $\square$

Lemma 9.2.

$$\mathcal{P}(M) \cong \bigoplus_{I} W(M, IM)$$

We now give the main theorem which relates this decomposition to the previous one:

Theorem 9.3.

$$W(M, IM) \cong \bigoplus_{\substack{N \subset M \\ M/N \cong \omega_I}} V(M, N)$$

First, we show the following lemma:

Lemma 9.4. *If $M/N \cong \omega_I$, then*

$$V(M,N) \subset W(M, IM)$$

Proof. (of Lemma 9.4) By assumption, $M/N \cong \omega_I$. I is the annihilator of ω_I. Therefore, N contains IM, but does not contain JM for any J that strictly contains I. $V(M, N)$ is contained in $\mathcal{P}(M, IM)$. It remains to prove the following claim:

Claim. *Suppose that the ideal J strictly contains I. Then $V(M,N)$ is orthogonal to $\mathcal{P}(M,JM)$.*

The claim can be deduced from the orthogonal decomposition:

$$\mathcal{P}(M,JM) \cong \bigoplus_{JM\subset N'} V(M,N')$$

$\square$

Deduction of Theorem 9.3 from Lemma 9.4 We have shown that

$$\bigoplus_{\substack{N\subset M\\ M/N\cong\omega_I}} V(M,N) \subset W(M,IM)$$

To prove the converse, we proceed by contradiction. Suppose that for some I, the inclusion is strict. We take the product over all I. It follows that

$$\bigoplus_{N\subset M} V(M,N) \cong \mathcal{P}(M)$$

is strictly contained in

$$\bigoplus_{I} W(M,IM) \cong \mathcal{P}(M)$$

This gives a contradiction.

10 An important inequality

Suppose that ν is a measure on M. Denote by ν_N the projection of ν on $V(M,N)$. Denote

$$|\nu_N| \stackrel{\text{def}}{=\!=} \Big|\Big|\nu_N\Big|\Big|_{L^1(M)},$$

the L^1 norm of ν_N. In a forthcoming article, on the universality of random matrices over R, we will need to bound the following quantity:

$$\frac{1}{|M/IM|}\sum_{\substack{N\subset M\\ M/N\cong\omega_I}} |\nu_N|$$

Definition. Denote by ($\nu \mod I$) the measure induced on M/IM by ν, via push-forward.

Remark. Alternatively, recall that $\mathrm{proj}_{IM}\nu$ is a measure in $\mathcal{P}(M,IM)$. Via the isomorphism $\mathcal{P}(M,IM) \cong \mathcal{P}(M/IM)$, $\mathrm{proj}_{IM}\nu$ defines a measure on M/IM. This measure is precisely ($\nu \mod I$).

Define

$$\Big|\Big| \cdot \Big|\Big|_{L^2(M/IM)} \stackrel{\text{def}}{=\!=} \Big\langle \cdot\,,\cdot \Big\rangle_{M/IM},$$

to be the usual Euclidean norm.

Theorem 10.1. *For any signed measure ν on M,*

$$\frac{1}{|M/IM|}\sum_{\substack{N\subset M\\ M/N\cong\omega_I}}|\nu_N|\leq\frac{1}{\sqrt{|R/I^*|}}\Big|\Big|\nu\mod I\Big|\Big|_{L^2(M/IM)}\tag{10.1}$$

Proof. We use the Cauchy-Schwarz inequality two times. ν_N is contant on N-cosets, therefore, since $M/N\cong\omega_I$, ν_N must be constant on IM-cosets. Therefore,

$$\Big|\Big|\nu_N\Big|\Big|_{L^1(M)}=\Big|\Big|\nu_N\mod I\Big|\Big|_{L^1(M/IM)}=$$
$$=\Big\langle\nu_N\mod I\,,\,\mathrm{sgn}(\nu_N\mod I)\Big\rangle_{M/IM}\leq$$
$$\leq\Big|\Big|\nu_N\mod I\Big|\Big|_{L^2(M/IM)}\sqrt{\#M/IM}$$

by Cauchy-Schwarz. Define

$$S\overset{\text{def}}{=\!=}\{N\subset M|M/N\cong\omega_I\}$$

The left hand side is

$$\frac{1}{|M/IM|}\sum_{N\in S}|\nu_N\mod I|\leq\frac{1}{\sqrt{|M/IM|}}\sum_{N\in S}\Big|\Big|\nu_N\mod I\Big|\Big|_{L^2(M/IM)}\leq$$
$$\leq\frac{\sqrt{\#S}}{\sqrt{|M/IM|}}\sqrt{\sum_{N\in S}\Big|\Big|\nu_N\mod I\Big|\Big|^2_{L^2(M/IM)}}\tag{10.2}$$

where the last equality follows again by Cauchy-Schwarz. Since the ν_N are orthogonal, we can rewrite the preceding expression as:

$$\sqrt{\frac{\#S}{|M/IM|}}\Big|\Big|\sum_{N\in S}\nu_N\mod I\Big|\Big|_{L^2(M/IM)}\leq\tag{10.3}$$

$$\leq\sqrt{\frac{\#S}{|M/IM|}}\Big|\Big|\sum_{IM\in N}\nu_N\mod I\Big|\Big|_{L^2(M/IM)}=\sqrt{\frac{\#S}{|M/IM|}}\Big|\Big|\nu\mod I\Big|\Big|_{L^2(M/IM)}$$

Finally, we note that

$$\sqrt{\frac{\#S}{|M/IM|}}\leq\sqrt{\frac{\#Sur(M,\omega_I)}{\#Hom(M,\omega_I)|R/I^*|}}\leq\frac{1}{|R/I^*|}$$

□

Remark. Although this does not substantially improve the bound, we remark that

$$\sum_{N\in S}\nu_N$$

is the projection of the measure ν on $W(M,IM)$. Hence, in Theorem 10.1, we can replace $(\nu\mod I)$ by the projection of $(\nu\mod I)$ onto $W(M/IM,0)$.

11 A uniform bound on the L^1 norm of a Fourier component

In our forthcoming paper on universality, we will need another inequality.

Lemma 11.1. *If ν is any probability measure, then*

$$\left|\left|\nu_\chi\right|\right|_{L^1(M)} \leq \sqrt{|im(\chi)|} \tag{11.1}$$

To show (11.1), we first make the following observation:

Claim.

$$\nu_\chi \mod ker(\chi)$$

is orthogonal to

$$(\nu_\chi - \nu) \mod ker(\chi).$$

Proof. (of claim) This follows from the construction of ν_χ. Indeed, by construction, ν_χ is orthogonal to $(\nu_\chi - \mathrm{proj}_{ker(\chi)}\nu)$. Now, by construction $\nu_\chi = \mathrm{proj}_{ker(\chi)}\nu_\chi$. Therefore,

$$\mathrm{proj}_{ker(\chi)}\nu_\chi$$

is orthogonal to

$$\mathrm{proj}_{ker(\chi)}(\nu_\chi - \nu)$$

The claim follows. □

Proof. (of Lemma 11.1)

$$\left|\left|\nu_\chi\right|\right|_{L^1(M)} = \left|\left|\mathrm{proj}_{ker(\chi)}\nu_\chi\right|\right|_{L^1(M)} = \left|\left|\nu_\chi \mod ker(\chi)\right|\right|_{L^1(M/ker(\chi))} \leq$$

$$\leq \sqrt{|M/ker(\chi)|}\left|\left|\nu_\chi \mod ker(\chi)\right|\right|_{L^2(M/ker(\chi))} \tag{11.2}$$

The last inequality follows by applying the Cauchy-Schwarz inequality. Now the preceding claim shows that

$$\left|\left|\nu_\chi \mod ker(\chi)\right|\right|_{L^2(M/ker(\chi))} \leq \left|\left|\nu \mod ker(\chi)\right|\right|_{L^2(M/ker(\chi))}$$

But the right hand side is the L^2 norm of a probability measure, and is therefore bounded above by 1. Hence (11.2) is bounded above by

$$\sqrt{|M/ker(\chi)|} = \sqrt{|im(\chi)|}.$$

□

References

[Eis95] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, New York, NY, 1995.