On the satisfaction frequency of spectral characterization conditions

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Spectral invariants

Definition

A spectral invariant of a graph Γ depends only on the spectrum (or the eigenvalues) of its adjacency matrix A_Γ. Examples:

- $\#$ of vertices of Γ.
- $\#$ of edges of Γ.
- $\#$ of loops of length k in Γ.

Question

Is the isomorphism type of Γ a spectral invariant?

Unfortunately, no, in general.

Generalized spectral invariants

Definition

A generalized spectral invariant of Γ depends only on the spectrum of its adjacency matrix A_Γ and the spectrum of $A_{\bar{\Gamma}}$. Examples:

$\#$ of loops of length k in Γ.

$\#$ of directed walks of length k in $\Gamma \stackrel{\text{def}}{=\!=} w_k$.

Question

Is the isomorphism type of Γ a generalized spectral invariant?

No, in general. But yes, under a certain condition on the number of walks.

The Walk Matrix

Definition

$$e \overset{\text{def}}{=\!=} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad W_\Gamma \overset{\text{def}}{=\!=} \left[\begin{array}{c|c|c|c|c} e & A_\Gamma e & A_\Gamma^2 e & \cdots & A_\Gamma^{n-1} e \end{array} \right]$$

Remark

$[W_\Gamma]_{ij}$ measures the number of directed walks of length $j - 1$ beginning at a given vertex i.

Facts about the walk matrix

Remark

$$W_\Gamma^T W_\Gamma = \begin{bmatrix} w_0 & w_1 & w_2 & \cdots & w_{n-1} \\ w_1 & w_2 & w_3 & & \\ w_2 & w_3 & w_4 & & \\ \vdots & & & \ddots & \\ w_{n-1} & & \cdots & & w_{2n-1} \end{bmatrix}$$

Corollary

Mod 2, the matrix

$$W_\Gamma^T W_\Gamma$$

has rank 0 *or* 1*, depending on the parity of* $w_0 = n$*. Hence,* $corank_{\mathbb{F}_2}(W_\Gamma) \geq \lfloor \frac{n}{2} \rfloor$.

Wang's condition

Theorem (Wang, 2017)

If $det(W_\Gamma)$ *is*

(A) minimally divisible by 2*, i.e.* $\left(\frac{det(W_\Gamma)}{2^{\lfloor n/2 \rfloor}}\right)$ *is odd,*

(B) not divisible by any odd square,

Then there is no other graph that has the same generalized spectrum as Γ*.*

Conjecture (Wang, 2017)

For Erdos-Renyi graphs, Wang's condition is satisfied with positive probability.

A linearization trick

Remark

Proving Wang's conjecture is a problem in Random Matrix Theory!

Remark

$$|det(W_\Gamma)| = \#coker(W_\Gamma) = \#coker \left[\; e \; \middle| \; A_\Gamma e \; \middle| \; \cdots \; \middle| \; A_\Gamma^{n-1} e \; \right] =$$

$$= \#coker \left[\; e \; \middle| \; A_\Gamma e \; \middle| \; \cdots \; \middle| \; A_\Gamma^{i} e \; \middle| \; \cdots \; \right] = \#coker_{\mathbb{Z}[x]}(A_\Gamma - xI|e)$$

Question

What is the distribution of $coker_{\mathbb{Z}[x]}(A_\Gamma - Ix|e)$?

(Similar in spirit to question of distribution of sandpile groups.)

Assumption $\#1$

Remark

The size of a module M is squarefree if and only if $\#(M/JM)$ is squarefree for all ideals J of finite index in $R = \mathbb{Z}[x]$.

Assumption ("Interchange of limits")

$$\lim_{n\to\infty} \inf_{J \text{ s.t. } R/J \text{ is finite}} \mathbb{P}(\#coker_{R/JR}(A_\Gamma - Ix|e) \text{ is odd-square-free }) =$$

$$= \inf_{J \text{ s.t. } R/J \text{ is finite}} \lim_{n\to\infty} \mathbb{P}\left(\#coker_{R/JR}(A_\Gamma - Ix|e) \text{ is odd-square-free }\right)$$

Second Assumption

Assumption ("Universality")

If $\#R/J$ is odd, then

$$coker_{R/J}\left[A_{\Gamma_n} - I_{n,n}x \middle| e\right] \text{ ias asymptotically distributed as:}$$

$$coker_{R/J}\left[SymUniform_{n,n}(R/J) \middle| Uniform_{n,1}(R/J)\right]$$

when $n \to \infty$.

Prediction for how often $det(W)$ is odd-square-free

Proposition

Under the two assumptions stated above,

$$\mathbb{P}(det(W) \text{ is odd-square-free }) =$$

$$\left[\prod_{p>2}\left(1-\frac{1}{p^2}-\frac{1}{p^3}+\frac{1}{p^4}\right)\right]\prod_{i=1}^{\infty}\zeta(2i) \sim$$

$$0.622\cdots$$

Main innovation

Remark

Our main innovation is an inductive method for understanding the distribution of cokernels of uniformly random matrices over finite rings, with or without symmetry, building on work of Evans.

Theorem (Evans)

Let $\mathcal{G}$ be the cokernel of an $n \times n$ random matrix, with entries uniformly random in $\mathbb{Z}/p^k\mathbb{Z}$. Then,

$$\mathbb{P}\left[\mathcal{G} \cong A\right] = \mathbb{P}\left[rank(\mathcal{G}) = rank(A)\right]\mathbb{P}\left[coker(pUniform_{rank(A),rank(A)}) \cong A\right]$$

"Probability splits". Evans uses this to show that $p^i\mathcal{G}/p^{i+1}\mathcal{G}$ is a Markov chain.

Main innovation, cont'd

Proposition (L.- Van Werde)

Let R be a finite local ring with maximal ideal $\mathfrak{m}$.

$$\mathbb{P}\left[coker(SymUniform_{n,n}(R)|Uniform_{n,1}(R) \cong A\right] =$$

$$= \mathbb{P}\left[corank(SymUniform_{n,n}(R)|Uniform_{n,1}(R)) = rank(A)\right] \times$$

$$\times \mathbb{P}\left[coker(SymUniform_{rank(A),rank(A)}(\mathfrak{m})|Uniform_{rank(A),1}(\mathfrak{m})) \cong A\right]$$

Other applications of this technique

Theorem (Fulman-Goldstein, 2015)

The corank distribution of uniformly random $n \times n$ matrices over $\mathbb{F}_p$ converges in total variation at rate $O\left(\frac{1}{p^n}\right)$. This holds for many symmetry types.

Corollary

Cokernels of uniformly random matrices over $\mathbb{Z}/p^n\mathbb{Z}$ converge at same rate, for the same symmetry types. $\mathbb{Z}/p^n\mathbb{Z}$ can be replaced by any finite local ring.

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